On Some 0- Conditions in the Lattice of Subgroups of the Symmetric Groups

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Abstract – In this paper we study about some 0-conditions in the lattice of subgroups of S_n. The 0-conditions in the lattice theoretic properties verified are: 0-distributivity, 0-modularity, 0-semi modularity, pseudo 0-distributivity, super 0-distributivity and 0-supermodularity in $L(S_n)$.

Index Terms – Symmetric Group, Lattice, 0-Conditions.

1. INTRODUCTION

In this paper we examine the 0-conditions in the subgroup lattices. R. Sulaiman, in his paper has given" Subgroups lattice of symmetric group S_4" in [8]. Subsequently we have given "The structure of the lattice of subgroups of the symmetric group S_5" in [14] and we have investigated there some basic properties satisfied by it. In this paper we study about the 0-conditions like 0-distributivity, 0-modularity, 0-semimodularity, pseudo 0-distributivity, super 0-distributivity and 0-supermodularity in $L(S_n)$, where $L(S_n)$ denote the subgroup lattice of S_n.

2. PRELIMINARY

We recall some lattice theoretic definitions that will be used later.

Definition 2.1

The symmetric group S_n is defined to be the group of all permutations on a set of n elements, ie, the symmetric group of degree n.

Definition 2.2

A Lattice L is said to be 0-distributive if for all x, y, z $\in$ L, whenever $x \wedge y = 0$ and $x \wedge z = 0$ then $x \wedge (y \vee z) = 0$.

Definition 2.3

A Lattice L is said to be 0-modular if whenever $x \leq y$ and $y \wedge z = 0$, then $x = (x \vee z) \wedge y$, for all x, y, z $\in$ L.

Definition 2.4

A Lattice L is said to be 0-semi modular if whenever a is an atom of L and x $\in$ L such that $a \wedge x = 0$, then $x \vee a$ covers x.

Definition 2.5

A lattice L is said to be pseudo 0-distributive if for all x, y, z $\in$ L with $x \wedge y = 0$, $x \wedge z = 0$ we have $(x \vee y) \wedge z = y \wedge z$.

Definition 2.6

A lattice L is said to be super 0-distributive if for all x, y, z $\in$ L, $x \wedge y = 0$ implies that $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

Definition 2.7

A lattice L is said to be 0-supermodular if for all a, b, c, d $\in$ L, with $b \wedge c = c \wedge d = b \wedge d = 0$, we have $(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a$.

We produce below the structure of the lattice of subgroups of the symmetric groups S_2, S_3, S_4 and S_5.

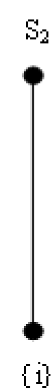

Fig. 2.1: Lattice Structure of $L(S_2)$

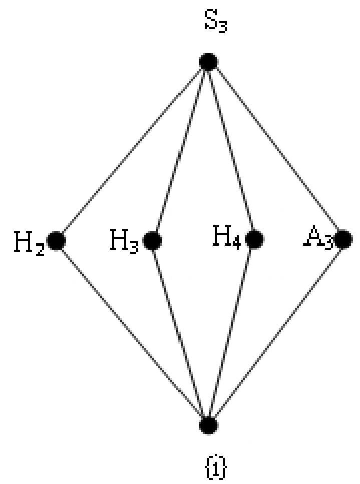

Fig. 2.2 : Lattice Structure of $L(S_3)$

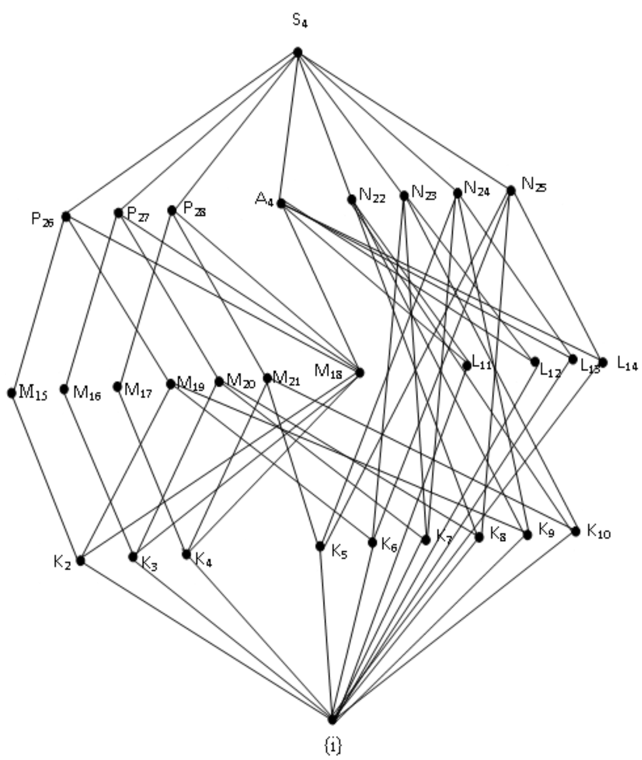

Fig. 2.3: Lattice Structure of $L(S_4)$ [8]

Row I : (Left to right): R_{129} to R_{138}, V_{150} to V_{154}, R_{139} to R_{143} and T_{144} to T_{149} as shown in Fig. 2.4.

Row II : (Left to right): N_{78} to N_{87}, N_{98} to N_{107},N_{88} to N_{97}, Q_{108} to Q_{122} and P_{123} to P_{128} as shown in Fig. 2.4.

Row III: (Left to right): K_{27} to K_{36}, L_{37} to L_{51}, L_{52} to L_{66}, L_{67} to L_{71} and M_{72} to M_{77} as shown in Fig. 2.4.

Row IV: (Left to right): H_2 to H_{11} and H_{12} to H_{26}. as shown in Fig. 2.4.

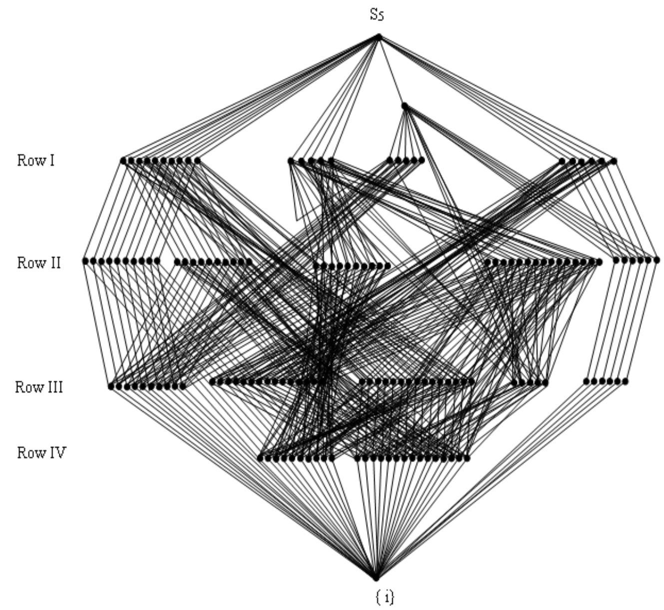

Fig. 2.4: Lattice Structure of $L(S_5)$ [14]

3. MAIN RESULT

0-conditions in the subgroup lattices of S_n

In this section we examine the 0-conditions in the subgroup lattices. The concept of 0-distributive lattices was first introduced by J.C.Varlet [9]. Several authors made contributions in different aspects of 0-distributive lattices. For example, one can refer to Y. S. Pawar [7], A. Vethamanickam [13], P. Balasubramanie [1], A. Veeramani [10] etc. 0-super modular lattices concept was first introduced by A. Vethamanickam and J. Arivukkarasu [11] as a lattice L in which $b \wedge c = c \wedge d = b \wedge d = 0$ implies $(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a$ for all a, b, c, d in L. Other conditions are 0- modularity, 0-semi modularity, 0-super modularity, etc. We examine them in $L(S_n)$. We observe that $L(S_i)$ is an interval of $L(S_n)$ for all $i<n$.

Lemma 3.1

$L(S_n)$ is 0-distributive if and only if $n \leq 2$

Proof:

Let $L(S_n)$ be 0-distributive

Suppose that $n > 2$, then $L(S_n)$ contains $L(S_3)$ as an interval which is not 0-distributive, since there are K_2, K_3 and K_4 in $L(S_3)$ such that $K_2 \wedge K_3 = \{i\}$ and $K_2 \wedge K_4 = \{i\}$, but $K_2 \wedge (K_3 \vee K_4) = K_2 \neq \{i\}$ Therefore, the 0-distributivity of $L(S_n)$ implies that $n \leq 2$.

Conversely, assume that $n \leq 2$ is true.

From Fig.2.1 and Fig.2.2 we see that $L(S_2)$ and $L(S_3)$ are 0-distributive. Hence the proof.

Lemma 3.2

$L(S_n)$ is 0-modular if and only if $n \leq 3$

Proof:

Let $L(S_n)$ be 0-modular

Suppose that $n > 3$, then $L(S_n)$ contains $L(S_4)$ as an interval which is not 0-modular, Since it contains the sublattice $\{\{i\}, K_4, M_{18}, L_{11}, A_4\}$ which is isomorphic to the pentagon N_5, which implies that $L(S_n)$ is not 0-modular.

Therefore, the 0-modularity of $L(S_n)$ implies that $n \leq 3$.

Conversely, assume that $n \leq 3$ is true.

From Fig.2.1 and Fig.2.2 we see that $L(S_2)$ and $L(S_3)$ are 0-modular, as they do not contain N_5 as a sublattice. Hence the proof.

Lemma: 3.3

$L(S_n)$ is 0-semimodular if and only if $n \leq 3$

Proof:

Let $L(S_n)$ be 0-semimodular

Suppose that $n > 3$, then $L(S_n)$ contains $L(S_4)$ as an interval which is not 0-semimodular. Since, we consider an atom K_4 and any element K_5 in $L(S_4)$ such that $K_4 \wedge K_5 = 0$ but $K_4 \vee K_5 = S_4$, which does not cover K_5. Therefore, $L(S_4)$ is not 0-semimodular. Therefore $L(S_n)$ is 0-semimodular implies that $n \leq 3$

Conversely, assume that $n \leq 3$ is true.

When $n = 2$, we know that $L(S_2)$ is isomorphic to B_1 , the 2-element chain, which is semi- modular. so it is 0-semi modular and $L(S_3)$ is isomorphic to M_4 , which is semi modular, so it is 0-semimodular. Hence the proof.

Lemma: 3.4

$L(S_n)$ is Pseudo 0-distributive if and only if $n \leq 2$

Proof:

Let $L(S_n)$ be Pseudo 0-distributive

Suppose $n > 2$, then $L(S_n)$ contains $L(S_3)$ as an interval which is not Pseudo 0-distributive, since, we consider the elements K_2, K_3 and K_4 in $L(S_3)$, such that $K_2 \wedge K_3=\{i\}$ and $K_2 \wedge K_4= \{i\}$, but $(K_2 \vee K_3) \wedge K_4 \neq K_3 \wedge K_4$

Therefore $L(S_n)$ is Pseudo 0-distributive implies that $n \leq 2$

Conversely, assume that $n \leq 2$ is true.

When $n = 2$, we know that $L(S_2)$ is isomorphic to B_1 , the 2-element chain, which is Pseudo distributive. So it is Pseudo 0-distributive. Hence the proof.

Lemma: 3.5

$L(S_n)$ is Super 0-distributive if and only if $n \leq 2$

Proof:

Let $L(S_n)$ be Super 0-distributive

Suppose $n > 2$, then $L(S_n)$ contains $L(S_3)$ as an interval which is not Super 0-distributive, since, we consider the elements K_2, K_3 and K_4 in $L(S_3)$, such that $K_2 \wedge K_3 = \{i\}$, but $(K_2 \vee K_3) \wedge K_4 \neq (K_2 \wedge K_4) \vee (K_3 \wedge K_4)$

Hence $L(S_n)$ is Super 0-distributive implies that $n \leq 2$

Conversely, assume that $n \leq 2$ is true.

When $n = 2$, we know that $L(S_2)$ is isomorphic to B_1 , the 2-element chain, which is super 0-distributive. Hence the proof.

Lemma: 3.6

$L(S_n)$ is 0-Supermodular if and only if $n \leq 2$

Proof:

Let $L(S_n)$ be 0-Supermodular

Suppose $n > 2$, then $L(S_n)$ contains $L(S_3)$ as an interval which is isomorphic to M_4 is not 0-Supermodular.

Hence $L(S_n)$ is 0-Supermodular implies that $n \leq 2$

Conversely, assume that $n \leq 2$ is true.

When $n = 2$, we know that $L(S_2)$ is isomorphic to B_1, the 2-element chain, which is 0-Supermodular. Hence the proof.

4. CONCLUSION

In this paper we have investigated some specified 0-conditions in the lattice $L(S_n)$, namely, 0-distributivity, 0-modularity, 0-semi modularity, pseudo 0-distributivity, super 0-distributivity and 0-super modularity.

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